JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **134**, 81–93 (1988)

A Monotone Iterative Scheme for Nonlinear Reaction-Diffusion Systems Having Nonmonotone Reaction Terms

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Submitted by V. Lakshmikantham

Received June 23, 1986

A monotone scheme is proposed for the solution of weakly coupled systems of reaction-diffusion equations without any monotonicity property of the nonlinear reaction terms. The considerations are taken within the framework of weak solutions. © 1988 Academic Press, Inc.

1. INTRODUCTION

Models in various fields of applications, such as biochemistry, biology, and chemical and nuclear engineering, can be described by systems of nonlinear parabolic initial-boundary-value problems of reaction-diffusion type. Constructive methods that yield not only existence results but also numerical procedures for the computation of solutions and error estimations are of great value.

The method of monotone iteration coupled with the notion of upper and lower solution or upper and lower quasi-solution as initial iterations, respectively, has been employed successfully by various authors to prove the existence of solutions or at least of quasi-solutions of nonlinear reaction-diffusion equations; cf. [2, 4, 5, 7, 9, 11–17, 20–22, 25]. Recently, in [23], the monotone method has been used for numerical computations.

Usually the monotone iterative technique only works if the nonlinear reaction terms possess at least a mixed quasi-monotone property; cf. the above-cited literature.

In this paper we establish a monotone scheme for systems of nonlinear reaction-diffusion equations for cases where the nonlinear terms do not possess any monotonicity property. By using suitable initial iterations we can construct monotone sequences as solutions of linear coupled systems that are related to the original problem in a certain sense. It can be shown

0022-247X/88 $3.00

that these sequences converge monotonically to a unique solution of the original nonlinear system from above and below, respectively.

Additionally the smoothness conditions on the data of the problem are weakened to a wide extent.

2. Definitions and Assumptions

We consider the nonlinear parabolic initial-bondary-value problem (IBVP for short)

$$\begin{aligned} \frac{\partial u_k}{\partial t} - \mathscr{L}_k u_k &= f_k(x, t, u) && \text{in } Q_T, \\ u_k(x, 0) &= \varphi_k(x) && \text{in } \Omega, \\ B_{k\delta} u_k &= 0 && \text{on } \Gamma_T, \end{aligned} \tag{1}$$

$k = 1, ..., n$, where $u \in R^n$, Ω is a bounded domain in R^N with a regular (Lipschitz continuous, [10, p. 31]) boundary $\partial\Omega$, $Q_T = (0, T) \times \Omega$, $\Gamma_T = (0, T) \times \partial\Omega$, $T > 0$.

For $\delta \in \{0, 1\}$ the boundary operators $B_{k\delta}$ are defined as follows $B_{k0} u_k := u_k$ and $B_{k1} := \partial u_k / \partial v^k + \beta^k(x, t)\, u_k$, where $\partial/\partial v^k$ denotes the outward conormal derivative on Γ_T. The uniformly elliptic operators $\mathscr{L}_k$ are of the form

$$\mathscr{L}_k := \frac{\partial}{\partial x_i}\left(a_{ij}^k(x, t)\frac{\partial}{\partial x_j}\right) - b_i^k(x, t)\frac{\partial}{\partial x_i},$$

where

$$a_{ij}^k(x, t)\, \xi_i \xi_j \geqslant \mu_k \xi_i^2,$$

for all real ξ_i and $(x, t) \in Q_T$.

Let the coefficients a_{ij}^k, b_i^k, and β^k be real, measurable, and bounded in their respective domains. It is assumed that $\beta^k(x, t) \geqslant 0$ and $\varphi_k \in L_2(\Omega)$ throughout this paper.

Let B be a Banach space, then we will denote by $B^n := B \times \cdots \times B$ the n-dimensional Cartesian product of B, which is again a Banach space endowed with the norm

$$\|(x_1, ..., x_n)\|_{B^n} := \sum_{i=1}^{n} \|x_i\|_B.$$

In this paper we generally consider weak solutions of (1) of the space W (cf. [18, p. 109]), defined by

$$W := \left\{ h \mid h \in V, \frac{\partial h}{\partial t} \in V' \right\},$$

where $\partial/\partial t$ denotes the distribution derivative in V',

$$V = L_2(0, T; W_2^1(\Omega)),$$
$$V' = \text{dual space of } V = L_2(0, T; (W_2^1(\Omega))'),$$

and

$$W_2^1(\Omega) = \left\{ h \mid h \in L_2(\Omega), \frac{\partial h}{\partial x_i} \in L_2(\Omega) \right\}$$

is the Sobolev space with its dual space $(W_2^1(\Omega))'$. The spaces V, V', and W of vector-valued functions are Banach spaces equipped with the norms

$$\|h\|_V^2 = \int_0^T \|h(\cdot, t)\|_{W_2^1(\Omega)}^2 \, dt,$$

$$\|h\|_{V'}^2 = \int_0^T \|h(\cdot, t)\|_{(W_2^1(\Omega))'}^2 \, dt,$$

$$\|h\|_W^2 = \|h\|_V^2 + \left\| \frac{\partial h}{\partial t} \right\|_{V'}^2.$$

By $\mathring{W}$, $\mathring{V}$, $\mathring{V}'$ we denote the coresponding spaces if the Sobolev space $W_2^1(\Omega)$ in the definitions of W, V, V' is replaced by its subspace $\mathring{W}_2^1$, i.e., the space of all functions of $W_2^1(\Omega)$ with zero traces on $\partial\Omega$; cf. [1].

According to the boundary operators $B_{k\delta}$ we introduce the bilinear form $l_{k\delta}$ defined by

$$l_{k\delta}(h, \chi) := \int_{Q_T} \left(a_{ij}^k \frac{\partial h}{\partial x_j} \frac{\partial \chi}{\partial x_i} + b_i^k \frac{\partial h}{\partial x_i} \chi \right) dx \, dt$$
$$+ \delta \int_{\Gamma_T} \beta^k h \chi \, ds \, dt.$$

Let $\langle \cdot, \cdot \rangle$ denote the scalar product of elements from V' and V.

DEFINITION. A function $u: Q_T \to R^n$ of $\mathring{W}^n$ (W^n) is called a *weak solution* of the IBVP (1), if the following conditions are fulfilled:

(i) $u(x, 0) = \varphi(x)$,

(ii) $\langle \partial u_k/\partial t, \chi \rangle + l_{k\delta}(u_k, \chi) = \int_{Q_T} f_k(x, t, u)\, \chi\, dx\, dt$, for all $\chi \in \mathring{V}(V)$ and for $\delta = 0$ $(\delta = 1)$, $k = 1, 2, ..., n$.

Hint. The condition (i) has to be taken in the sense

$$\lim_{t \to 0} \| u_k(\cdot, t) - \varphi_k \|_{L_2(\Omega)} = 0, \qquad \text{for all } k = 1, 2, ..., n.$$

This is well defined because of the continuous embedding $W \hookrightarrow C([0, T]; L_2(\Omega))$ (cf. [18, p. 110, Theorem 1.1.]), that will also be used in later chapters.

Let us denote by $[u, v]$ the order interval in $(L_2(Q_T))^n$, that is, $[u, v] := \{z \in (L_2(Q_T))^n \mid u_k \leqslant z_k \leqslant v_k$, for $k = 1, ..., n$ and almost everywhere in $Q_T\}$. Further we define the vector $[u]_k \in R^{n-1}$ for $u \in R^n$ by

$$[u]_k := (u_1, ..., u_{k-1}, u_{k+1}, ..., u_n).$$

Let $M = (M_{ij})$, $i, j = 1, ..., n$, be a matrix, then we denote by $[M]_k \in R^{n-1}$ the vector

$$[M]_k := (M_{k1}, ..., M_{k,k-1}, M_{k,k+1}, ..., M_{kn}).$$

Using the definitions made above, the right-hand sides $f_k(x, t, u)$ of (1) can be rewritten in the form $f_k(x, t, u_k, [u]_k)$. Finally, we define functions $\tilde{f}_k$ for $u, v \in R^n$ by

$$\tilde{f}_k(x, t, u_k, [v]_k) := f(x, t, u_k, [v]_k) - [M]_k\, [v]_k.$$

Let I be some order interval in $(L_2(Q_T))^n$, then the following hypotheses will be used in the next chapter:

(H1) The right-hand sides f_k are of Carathéodory type and satisfy for $u, v \in I$ a uniform Lipschitz condition of the form

$$|(f_k(\cdot, \cdot, u) - f_k(\cdot, \cdot, v)| \leqslant \gamma_{ki}\, |u_i - v_i|.$$

(H2) There exists a matrix $M = (M_{ij}) \geqslant 0$ and vectors ϕ, $\psi \in W^n$, such that the following inequalities are fulfilled:

$$\begin{aligned} \frac{\partial \phi_k}{\partial t} - \mathscr{L}_k \phi_k - [M]_k [\phi]_k &\geqslant \tilde{f}_k(x, t, \phi_k, [\psi]_k), \\ \phi_k(x, 0) &\geqslant \varphi_k(x), \\ B_{k\delta} \phi_k &\geqslant 0 \quad \text{on} \quad \Gamma_T, \end{aligned} \tag{2}$$

and

$$\frac{\partial \psi_k}{\partial t} - \mathscr{L}_k \psi_k - [M]_k [\psi]_k \leqslant \tilde{f}_k(x, t, \psi_k, [\phi]_k)$$
$$\psi_k(x, 0) \leqslant \varphi_k(x), \tag{3}$$
$$B_{k\delta} \psi_k \leqslant 0 \qquad \text{on} \quad \Gamma_T,$$

where "$[\cdot]_k [\cdot]_k$" is the scalar product in R^{n-1} and the constant matrix M has to be chosen in such a way that the right-hand sides $\tilde{f}_k$ are quasi-monotone nonincreasing in I. The latter can always be obtained due to hypothesis (H1).

Hint. Notice that the inequalities (2) and (3) have to be understood in an appropriate weak sense and that the summation convention is used.

3. A Comparison Result

An important tool for our monotone scheme is the following comparison result.

THEOREM 1. *Let the hypotheses* (H1) *and* (H2) *be satisfied for a pair of functions* ϕ, $\psi \in W^n$ *with respect to the order interval* $I := [\inf(\phi, \psi), \sup(\phi, \psi)]$. *Then* $\psi \leqslant \phi$.

Proof. The proof will be given only for Dirichlet boundary condition, i.e., $\delta = 0$, because the proof for the case $\delta = 1$ follows essentially the same arguments.

Due to hypothesis (H2) the functions ϕ and ψ fulfill inequalities (2) and (3), respectively, in the weak sense

$$\left\langle \frac{\partial \phi_k}{\partial t}, \chi \right\rangle + l_{k0}(\phi_k, \chi) - \int_{Q_T} [M]_k [\phi]_k \chi \, dx \, dt$$
$$\geqslant \int_{Q_T} \tilde{f}_k(x, t, \phi_k, [\psi]_k) \chi \, dx \, dt,$$
$$\left\langle \frac{\partial \psi_k}{\partial t}, \chi \right\rangle + l_{ko}(\psi_k, \chi) - \int_{Q_T} [M]_k [\psi]_k \chi \, dx \, dt$$
$$\leqslant \int_{Q_T} \tilde{f}_k(x, t, \psi_k, [\phi]_k) \chi \, dx \, dt,$$

for all test functions $\chi \in \mathring{V}$, $\chi \geqslant 0$, and

$$\psi_k(x, 0) \leqslant \varphi_k(x) \leqslant \phi_k(x, 0),$$
$$\psi_k(x, t) \leqslant 0 \leqslant \phi_k(x, t) \qquad \text{on} \quad \Gamma_T.$$

By subtracting corresponding inequalities from each other we get the following inequalities for the auxiliary function $w_k := \psi_k - \phi_k$:

$$\left\langle \frac{\partial w_k}{\partial t}, \chi \right\rangle + l_{k0}(w_k, \chi) - \int_{Q_T} [M]_k\, [w]_k\, \chi\, dx\, dt.$$
$$\leqslant \int_{Q_T} (\tilde{f}_k(x, t, \psi_k, [\phi]_k) - \tilde{f}_k(x, t, \phi_k, [\psi]_k))\, \chi\, dx\, dt. \tag{4}$$

Now, as a special test function we choose $\chi = w_k^+$, where w_k^+ is defined by $w_k^+ := \max_{Q_T}(w_k, 0)$.

Notice that because of $w_k(x, t) \leqslant 0$ on Γ_T the functon $\chi = w_k^+$ is an admissible test function, i.e., $\chi \in \mathring{V}$ and $\chi \geqslant 0$; cf. [6]. Taking into account the quasi-monotonicity and the Lipschitz continuity of the functions $\tilde{f}_k$ we can estimate the right-hand side of inequality (4) as

$$\int_{Q_T} (\tilde{f}_k(x, t, \psi_k, [\phi]_k) - \tilde{f}_k(x, t, \phi_k, [\psi]_k))\, w_k^+\, dx\, dt$$
$$= \int_{Q_T} (\tilde{f}_k(x, t, \psi_k, [\phi]_k) - \tilde{f}_k(x, t, \phi_k, [\phi]_k))\, w_k^+\, dx\, dt$$
$$+ \int_{Q_T} (\tilde{f}_k(x, t, \phi_k, [\phi]_k) - \tilde{f}_k(x, t, \phi_k, [\psi]_k))\, w_k^+\, dx\, dt$$
$$\leqslant \int_{Q_T} \{\gamma_{kk} |\phi_k - \psi_k|\, w_k^+ + [\gamma + M]_k\, [(\psi - \phi)^+]_k\, w_k^+\}\, dx\, dt$$
$$\leqslant \int_{Q_T} \{\gamma_{kk}(w_k^+)^2 + [\gamma + M]_k\, [w^+]_k\, w_k^+\}\, dx\, dt. \tag{5}$$

Next, it can easily be shown that

$$\int_{Q_T} [M]_k\, [w]_k\, w_k^+\, dx\, dt \leqslant \int_{Q_T} [M]_k\, [w^+]_k\, w_k^+\, dx\, dt, \tag{6}$$
$$l_{k0}(w_k, w_k^+) = l_{k0}(w_k^+, w_k^+), \qquad \text{cf. [6].} \tag{7}$$

Because of $w_k(x, 0) \leqslant 0$ we have

$$\left\langle \frac{\partial w_k}{\partial t}, w_k^+ \right\rangle = \frac{1}{2} \int_\Omega (w_k^+(\cdot, T))^2\, dx. \tag{8}$$

The latter can be deduced from [19, p. 304]. With (5), (6), (7), and (8) we get from (4) the estimation

$$\frac{1}{2}\int_{\Omega} (w_k^+(\cdot, T))^2\,dx + l_{k0}(w_k^+, w_k^+)$$
$$\leqslant \int_{Q_T} \{\gamma_{kk}(w_k^+)^2 + [2M+\gamma]_k\,[w^+]_k\,w_k^+\}\,dx\,dt. \tag{9}$$

For sufficiently large λ it is always possible to obtain that

$$l_{k0}(w_k^+, w_k^+) + \lambda\int_{Q_T} (w_k^+)^2\,dx\,dt \geqslant 0, \tag{10}$$

for $k = 1, 2, ..., n$. Thus, from (9) it follows that

$$\int_{\Omega} (w_k^+(\cdot, T))^2\,dx \leqslant \int_{Q_T} 2\{(\lambda+\gamma_{kk})(w_k^+)^2 + [2M+\gamma]_k\,[w^+]_k\,w_k^+\}\,dx\,dt$$
$$\leqslant c\int_{Q_T}\sum_{j=1}^{n} (w_j^+)^2\,dx\,dt. \tag{11}$$

Summing up all the inequalities (11) from $k=1$ to $k=n$ and setting

$$y(T) := \int_{Q_T}\sum_{j=1}^{n} (w_j^+)^2\,dx\,dt,$$

we receive an ordinary differential inequality in y of the form

$$\frac{dy}{dT} \leqslant cy, \tag{12}$$

where $y(0)=0$ and $y(T)\geqslant 0$ for $T\geqslant 0$.

The application of Gronwall's lemma immediately yields $y(T)\equiv 0$ for $T\geqslant 0$. The latter means $w_k^+\equiv 0$ for all k and hence it follows that $\psi_k\leqslant\phi_k$. This completes the proof.

4. Preliminaries

The proof of the monotone iterative technique in Section 5 requires an existence and uniqueness result for linear coupled systems of the form

$$\frac{\partial u_k}{\partial t} - \mathscr{L}_k u_k + Q_{kj}u_j = G_k \qquad \text{in} \quad Q_T,$$
$$u_k(x, 0) = \varphi_k(x) \qquad \text{in} \quad \Omega, \tag{13}$$
$$B_{k\delta}u_k = 0 \qquad \text{on} \quad \Gamma_T,$$

where $G_k \in \mathring{V}'(V')$ and $Q = (Q_{ij})$ is an arbitrary matrix, the elements of which are bounded and measurable functions. Here we formulate a result which can be deduced from [18, Chap. 3].

THEOREM 2. *Let the assumptions of Section* 2 *on the data of the IBVP* (13) *be satisfied. Then there exists a uniquely defined solution* $u \in W^n$ *of* (13) *and the following estimation holds*:

$$\| u \|^2_{W^n} \leqslant c\,(\| \varphi \|^2_{(L_2(\Omega))^n} + \| G \|^2_{(V')^n}).$$

Notice that if in the special case $G_k = G_k(x, t) \in L_2(Q_T)$ then because of the continuous embedding $L_2(Q_T) \hookrightarrow \mathring{V}'(V')$ we get an estimation of the form

$$\| u \|^2_{W^n} \leqslant c\,(\| \varphi \|^2_{(L_2(\Omega))^n} + \| G \|^2_{(L_2(Q_T))^n}). \tag{14}$$

Now for u, $v \in R^n$ we introduce the function F_k defined by $F_k(x, t, u_k, [v]_k) := \tilde{f}_k(x, t, u_k, [v]_k) + M_{kk} u_k$. In addition to the hypotheses (H1) and (H2) we pose the following one:

(H3) Let the function $F_k = F_k(x, t, u_k, [v]_k)$ be monotone nondecreasing with respect to the component u_k within the order interval I.

The hypothesis (H3) can always be fulfilled due to hypothesis (H1).

5. MONOTONE ITERATIVE TECHNIQUE

The main result of this section can be formulated as follows.

THEOREM 3. *Let hypotheses* (H1), (H2), *and* (H3) *be satisfied for a pair of funcons* ϕ, $\psi \in W^n$ *with respect to the order interval* $I := [\inf(\psi, \phi), \sup(\psi, \phi)]$. *Then the IBVP* (1) *possesses a uniquely determined solution* $u \in W^n$ *with* $\psi \leqslant u \leqslant \phi$. *Moreover one can construct sequences* $\{u^i\}$ *and* $\{v^i\}$ *that converge monotonically* (*in* $(L_2(Q_T))^n$) *to the solution from above and below, respectively.*

Proof. The proof is based on the iterative scheme

$$\begin{aligned} &\frac{\partial u_k^{i+1}}{\partial t} - \mathscr{L}_k u_k^{i+1} + M_{kk} u_k^{i+1} - [M]_k\,[u^{i+1}]_k = F_k(x, t, u_k^i, [v^i]_k), \\ &\frac{\partial v_k^{i+1}}{\partial t} - \mathscr{L}_k v_k^{i+1} + M_{kk} v_k^{i+1} - [M]_k\,[v^{i+1}]_k = F_k(x, t, v_k^i, [u^i]_k), \end{aligned} \tag{15}$$

$$u_k^{i+1}(x, 0) = v_k^{i+1}(x, 0) = \varphi_k(x),$$
$$B_{k\delta} u_k^{i+1} = B_{k\delta} v_k^{i+1} = 0 \qquad \text{on} \quad \Gamma_T,\ k = 1, 2, ..., n.$$

In the first part it will be shown that starting with initial iterates $u^0 = \phi$ and $v^0 = \psi$ we get a monotone nonincreasing sequence $\{u^i\}$ and a monotone nondecreasing sequence $\{v^i\}$, respectively, and

$$\psi \leqslant v^i \leqslant u^i \leqslant \phi. \tag{16}$$

In the second part the convergence of these sequences to the uniquely determined solution u will be proved.

(a) As a result of Theorem 1 we immediately have $v^0 = \psi \leqslant \phi = u^0$. Now, let us show that $v^1 \leqslant u^1$. The difference $w^1 := v^1 - u^1$ satisfies the linear IBVP

$$\frac{\partial w_k^1}{\partial t} - \mathscr{L}_k w_k^1 + M_{kk} w_k^1 - [M]_k [w^1]_k$$

$$= F_k(x, t, v_k^0, [u^0]_k) - F_k(x, t, u_k^0, [v^0]_k),$$

with initial and boundary values $w_k^1(x, 0) = 0$ and $B_{k\delta} w_k^1 = 0$ on Γ_T. Since the right-hand side of the last equation is nonpositive, we deduce from Theorem 1 that $w_k^1 \leqslant 0$, i.e., $v^1 \leqslant u^1$. By an induction argument we can show analogously that $v^i \leqslant u^i$ for all $i = 1, 2, \ldots$.

Now we shall show the monotonicity behaviour of the sequences $\{u^i\}$ and $\{v^i\}$. By means of the iterative scheme (15) we get the following inequalities for the functions $w^1 := u^1 - u^0$ and $z^1 := v^0 - v^1$:

$$\frac{\partial w_k^1}{\partial t} - \mathscr{L}_k w_k^1 + M_{kk} w_k^1 - [M]_k [w^1]_k \leqslant 0,$$

$$\frac{\partial z_k^1}{\partial t} - \mathscr{L}_k z_k^1 + M_{kk} z_k^1 - [M]_k [z^1]_k \leqslant 0,$$

$w_k^1(x, 0) \leqslant 0$, $z_k^1(x, 0) \leqslant 0$, $B_{k\delta} w_k^1 \leqslant 0$, and $B_{k\delta} z_k^1 \leqslant 0$ on Γ_T. Hence, $w^1 \leqslant 0$ and $z^1 \leqslant 0$ due to Theorem 1, i.e., $u^1 \leqslant u^0$ and $v^0 \leqslant v^1$.

Now, it is assumed that the inequalities $u^i \leqslant u^{i-1}$ and $v^i \geqslant v^{i-1}$ are fulfilled. Again, from our iterative scheme (15), we obtain the following IBVP for the difference $w^{i+1} := u^{i+1} - u^i$ and $z^{i+1} := v^i - v^{i+1}$:

$$\frac{\partial w_k^{i+1}}{\partial t} - \mathscr{L}_k w_k^{i+1} + M_{kk} w_k^{i+1} - [M]_k [w^{i+1}]_k$$

$$= F_k(x, t, u_k^i, [v^i]_k) - F_k(x, t, u_k^{i-1}, [v^{i-1}]_k),$$

$$\frac{\partial z_k^{i+1}}{\partial t} - \mathscr{L}_k z_k^{i+1} + M_{kk} z_k^{i+1} - [M]_k [z^{i+1}]_k$$
$$= F_k(x, t, v_k^{i-1}, [u^{i-1}]_k) - F_k(x, t, v_k^i, [u^i]_k),$$
$$w_k^{i+1}(x, 0) = z_k^{i+1}(x, 0) = 0$$

and

$$B_{k\delta} w_k^{i+1} = B_{k\delta} z_k^{i+1} = 0 \qquad \text{On} \quad \Gamma_T.$$

Because of the monotonicity behaviour of the functions F_k due to hypotheses (H2) and (H3), the right-hand sides of the equations in w_k^{i+1} and z_k^{i+1}, respectively, are nonpositive. Hence, due to Theorem 1 the solutions w_k^{i+1} and z_k^{i+1} are non–positive too, i.e., $u^{i+1} \leqslant u^i$ and $v^i \leqslant v^{i+1}$.

(b) Owing to hypothesis (H1) the right-hand sides F_k of (15) can be regarded as continuous and bounded operators from $I \subset (L_2(Q_T))^n$ into $L_2(Q_T)$. Applying Theorem 2 and the estimation (14) we obtain for the uniquely determined solutions u^{i+1}, $v^{i+1} \in W^n$ of (15) the estimations

$$\begin{aligned} \|u^{i+1}\|_{W^n}^2 &\leqslant c\left(\|\varphi\|_{(L_2(\Omega))^n}^2 + \sum_{k=1}^n \|F_k(\cdot, \cdot, u_k^i, [v^i]_k)\|_{L_2(Q_T)}^2\right), \\ \|v^{i+1}\|_{W^n}^2 &\leqslant c\left(\|\varphi\|_{(L_2(\Omega))^n}^2 + \sum_{k=1}^n \|F_k(\cdot, \cdot, v_k^i, [u^i]_k)\|_{L_2(Q_T)}^2\right). \end{aligned} \tag{17}$$

The right-hand sides of (17) are bounded for all i, thus the sequences $\{u^i\}$ and $\{v^i\}$ are bounded in W^n. Because of the weak compactness of a ball in a reflexive Banach space there exist subsequences of $\{u^i\}$ and $\{v^i\}$, which are weakly convergent in W^n. Further, the compact embedding $W^n \subsetneq (L_2(Q_T))^n$ implies the convergence of some subsequences of $\{u^i\}$ and $\{v^i\}$ in $(L_2(Q_T))^n$, and thus, due to the monotonicity behaviour of the iterates u^i and v^i, the whole sequences $\{u^i\}$ and $\{v^i\}$ must be convergent in $(L_2(Q_T))^n$ to u and v, respectively. From [10, p. 10, Lemma 5.4] the weak convergence in W^n of the whole sequences $\{u^i\}$ and $\{v^i\}$ to u and v, respectively, can be deduced. Now, the limit process ($i \to \infty$) can be carried out in the weak formulation of the IBVP (15). Hence it follows that the limits u and v are solutions of the coupled IBVP

$$\begin{aligned} &\frac{\partial u_k}{\partial t} - \mathscr{L}_k u_k + M_{kk} u_k - [M]_k [u]_k = F_k(x, t, u_k, [v]_k), \\ &\frac{\partial v_k}{\partial t} - \mathscr{L}_k v_k + M_{kk} v_k - [M]_k [v]_k = F_k(x, t, v_k, [u]_k), \end{aligned} \tag{18}$$

$$u_k(x, 0) = v_k(x, 0) = \varphi_k(x)$$

and

$$B_{k\delta} u_k = B_{k\delta} v_k = 0 \qquad \text{on} \quad \Gamma_T.$$

From inequality (16) one immediately gets $v \leqslant u$. On the other hand, u and v satisfy inequalities (3) and (2) of hypothesis (H2), respectively, and hence it follows from Theorem 1 that $u \leqslant v$. Thus we get $u = v$ and from (18) we deduce that u is a solution of the original IBVP (1). Further, from (16) follows $\psi \leqslant v^i \leqslant u \leqslant u^i \leqslant \phi$. By means of hypothesis (H1) the uniqueness of the solution u within the order interval I can easily be shown. This completes the proof.

Remark. (i) As has been shown, the iterative scheme (15) yields a monotone enclosure of the solution $u \in I$ of the IBVP (1) by monotone sequences $\{u^i\}$ and $\{v^i\}$, each of which converges to u in $(L_2(Q_T))^n$. Each iterative step requires the solution of two n-dimensional coupled systems of linear equations.

(ii) Of course, Theorem 3 can also be interpreted as an invariant set result. One can show that the hypothesis (H2) implies the "tangency condition," which was employed by various authers (cf. [3, 8, 24, 26]) to get invariant set results. But here we obtain in addition a monotone procedure for constructing the solution and the Lipschitz continuity of the f_k is only needed for the interval I.

(iii) The crucial point for establishing such a monotone iterative procedure is hypothesis (H2), i.e., to find a pair of functions ϕ, $\psi \in W^n$ which satisfy inequalities (2) and (3). In the last section we give an example where the initial iterates $v^0 = \psi$ and $u^0 = \phi$ can be easily constructed.

6. Example

Let the following IBVP be given:

$$\frac{\partial u_1}{\partial t} - \Delta u_1 = \cos u_1 \sin u_2,$$

$$\frac{\partial u_2}{\partial t} - \Delta u_2 = \sin u_1 \cos u_2,$$

$$u_k(x, 0) = \varphi_k(x), \qquad u_k(x, t) = 0 \qquad \text{on} \quad \Gamma_T, k = 1, 2.$$

Let $0 \leqslant \varphi_k(x) \leqslant K$ be assumed.

Using the denotation of the previous sections we have

$$f_1(u_1, u_2) = \cos u_1 \sin u_2,$$
$$f_2(u_1, u_2) = \sin u_1 \cos u_2.$$

To fulfill hypothesis (H2) we have to find functions ψ and ϕ, such that the following inequalities are satisfied:

$$\frac{\partial \phi_1}{\partial t} - \Delta\phi_1 - M_{12}\phi_2 \geqslant f_1(\phi_1, \psi_2) - M_{12}\psi_2,$$

$$\frac{\partial \phi_2}{\partial t} - \Delta\phi_2 - M_{21}\phi_1 \geqslant f_2(\psi_1, \phi_2) - M_{21}\psi_1,$$

$$\frac{\partial \psi_1}{\partial t} - \Delta\psi_1 - M_{12}\psi_2 \leqslant f_1(\psi_1, \phi_2) - M_{12}\phi_2,$$

$$\frac{\partial \psi_2}{\partial t} - \Delta\psi_2 - M_{21}\psi_1 \leqslant f_2(\phi_1, \psi_2) - M_{21}\phi_1$$

$$\psi_k(x, 0) \leqslant \varphi_k(x) \leqslant \phi_k(x, 0),$$
$$\psi_k(x, t) \leqslant 0 \leqslant \phi_k(x, t) \qquad \text{on} \quad \Gamma_T,$$

where the right-hand sides have to be quasi-monotone nonincreasing.

To get a solution of the above inequalities we take $M_{12} = M_{21} = 1$ and set $\phi_k = -\psi_k = \varepsilon e^{\lambda t}$, $k = 1, 2$, where the parameters ε and λ are to be chosen appropriately. By an elementary calculation one finds that $\phi_k = -\psi_k = \varepsilon e^{\lambda t}$, $k = 1, 2$, is a solution of the inequalities above, if $\varepsilon \geqslant \max(1, K)$ and $\lambda \geqslant 3$. Notice that in the order interval $I = [\psi, \phi]$ the nonlinearities f_1 and f_2 are not mixed quasi-monotone, provided that K is sufficiently large. But nevertheless we do have a monotone iterative procedure of the form (15), because all the suppositions of Theorem 3 are satisfied.

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